Crack the Problem (59): All the roots of the equation $x^6 - 6x^5 + 15x^4 - ax^3 + 15x^2 - bx + 1 = 0$ are positive. Find a and b.

Solution

The equation is $x^6 - 6x^5 + 15x^4 - ax^3 + 15x^2 - bx + 1 = 0$.
Let it has A, B, C, D, E & F as its roots, which are all positive.

Then, from the equation we can write
A + B + C + D + E + F = - (coefficient of x^5) / (coefficient of x^6) = 6/1 = 6 …… (i)
AB + BC + CA + …….. up to 15 terms = ∑AB = (coefficient of x^4) / (coefficient of x^6) = 15/1 = 15 ….. (ii)
ABCD + BCDE + …… up to 15 terms = ∑ABCD = (coefficient of x^2) / (coefficient of x^6) = 15/1 = 15 …. (iii)
ABCDEF = (constant term) / (coefficient of x^6) = 1/1 = 1 …. (iv)
Now,
By AM – GM inequality
AM ≥ GM and equality holds only when terms of AM are all equal.
Applying AM – GM inequality on the roots

$$\frac{A+B+C+D+E+F}{6} \geq \sqrt[6]{ABCDEF}$$

$$\Rightarrow \frac{6}{6} \geq \sqrt[6]{1} \Rightarrow 1 \geq 1$$ **(from (i) and (iv))**

Thus both sides are equal.
But, equality holds only when terms of AM are all equal.
So, A = B = C = D = E = F ……… (v)

From (i) and (v)
6A = 6 ⇒ A = 1
⇒ A = B = C = D = E = F = 1

That is the equation has six repeated roots i.e. 1
This also satisfies (i) , (ii) , (iii) and (iv) .
So the equation is actually $(x-1)^6 = 0$

Hence $(x-1)^6 = x^6 - 6x^5 + 15x^4 - ax^3 + 15x^2 - bx + 1$
$\Rightarrow x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1 = x^6 - 6x^5 + 15x^4 - ax^3 + 15x^2 - bx + 1$
comparing same coefficients we get a = 20 and b = 6

So a = 20 , b = 6.

Alternate Solution

The equation is $x^6 - 6x^5 + 15x^4 - ax^3 + 15x^2 - bx + 1 = 0$.

Let it has A, B, C, D, E & F as its roots, which are all positive.

Then, from the equation we can write
AB + BC + CA + …….. up to 15 terms = ∑AB = (coefficient of x^4) / (coefficient of x^6) = 15/1 = 15 ….. (i)
ABCD + BCDE + …… up to 15 terms = ∑ABCD = (coefficient of x^2) / (coefficient of x^6) = 15/1 = 15 ….. (ii)
ABCDEF = (constant term) / (coefficient of x^6) = 1/1 = 1 …. (iii)

Now,

$$\left(\frac{1}{AB}+\frac{1}{BC}+\frac{1}{CA}+........upto..15..terms\right)=\frac{CDEF+ADEF+BDEF+..........}{ABCDEF}=\frac{\Sigma ABCD}{ABCDEF}=\frac{15}{1}=15 \quad .... (iv)$$

(from (ii) and (iii))

Now, adding (i) and (iv) we get

$$\left(AB+\frac{1}{AB}\right)+\left(BC+\frac{1}{BC}\right)+\left(CA+\frac{1}{CA}\right)+....upto....15...terms....=30$$

$$\Rightarrow\left(AB+\frac{1}{AB}-2\right)+\left(BC+\frac{1}{BC}-2\right)+\left(CA+\frac{1}{CA}-2\right)+....upto....15...terms....=0$$

$$\Rightarrow$$

$$\left((\sqrt{AB})^2+\frac{1}{(\sqrt{AB})^2}-2\sqrt{AB}.\frac{1}{\sqrt{AB}}\right)+\left((\sqrt{BC})^2+\frac{1}{(\sqrt{BC})^2}-2\sqrt{BC}.\frac{1}{\sqrt{BC}}\right)+\left((\sqrt{CA})^2+\frac{1}{(\sqrt{CA})^2}-2\sqrt{CA}\frac{1}{\sqrt{CA}}\right)+...upto...15..terms...=0$$

(since roots are positive so $\sqrt{AB},\sqrt{BC}....$ all exist)

$$\Rightarrow\left(\sqrt{AB}-\frac{1}{\sqrt{AB}}\right)^2+\left(\sqrt{BC}-\frac{1}{\sqrt{BC}}\right)^2+\left(\sqrt{CA}-\frac{1}{\sqrt{CA}}\right)^2+.....upto...15...terms...=0 \quad .... (v)$$

Now, the sum of squares is 0 only when the individual terms are 0.

So, from (v) we get

$$\left(\sqrt{AB}-\frac{1}{\sqrt{AB}}\right)=0,..\left(\sqrt{BC}-\frac{1}{\sqrt{BC}}\right)=0,..\left(\sqrt{CA}-\frac{1}{\sqrt{CA}}\right)=0..................$$

$$\Rightarrow\left(\sqrt{AB}=\frac{1}{\sqrt{AB}}\right)...\left(\sqrt{BC}=\frac{1}{\sqrt{BC}}\right)...\left(\sqrt{CA}=\frac{1}{\sqrt{CA}}\right)..................$$

⇒AB = 1 , BC = 1 , CA = 1 , AD = 1 , …………. And so on
⇒ A = 1/B and C = 1/B
⇒ A = C
Similarly we will get A = B = C = D = E = F
Now , AB = 1 ⇒ A^2 = 1 ⇒ A = 1 or –1 ⇒ A = 1 (as roots are positive)
So A = B = C = D = E = F = 1
That is the equation has six repeated roots i.e. 1

So the equation is actually $(x-1)^6 = 0$

Hence $(x-1)^6 = x^6 - 6x^5 + 15x^4 - ax^3 + 15x^2 - bx + 1$
$\Rightarrow x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1 = x^6 - 6x^5 + 15x^4 - ax^3 + 15x^2 - bx + 1$
comparing same coefficients we get a = 20 and b = 6

So a = 20 , b = 6.

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